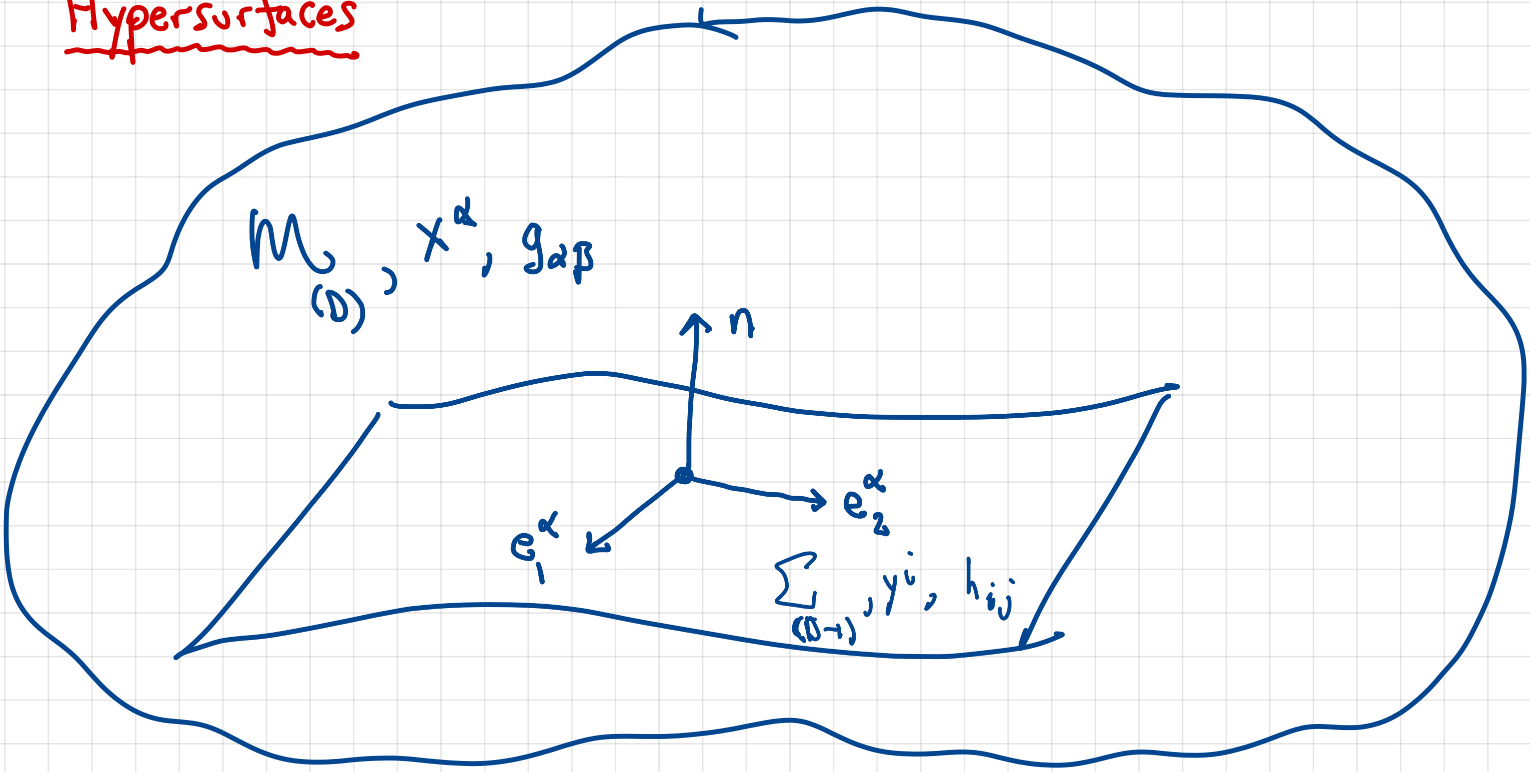
8. Proof of the Area Theorem

Read: Wald § 9.2

More refs –

Carroll appendix D
Poisson § 2, 3

Hypersurfaces



Normal n^α, $\quad n^2 = \pm 1$

Projector

$$e_i^\alpha = \left.\frac{\partial x^\alpha}{\partial y^i}\right|_\Sigma , \qquad e_i^\alpha n_\alpha = 0$$

Suppose Σ spacelike, $n^2 = -1$

Decompose metric

$$g_{\alpha\beta} = \underbrace{h_{\alpha\beta}}_{\text{space}} - \underbrace{n_\alpha n_\beta}_{\text{time}}$$

h is transverse:

$$n^\alpha h_{\alpha\beta} = n_\beta + n^2 n_\beta = 0$$

So $h_\alpha{}^\beta$ = projector: $\quad n^\alpha \left(h_\alpha{}^\beta v_\beta\right) = 0$ etc.

The <u>induced</u> <u>metric</u> on Σ is

$$h_{ij} = e_i^{\alpha} e_j^{\beta} g_{\alpha\beta}$$

check:

$$\begin{aligned} ds^2\big|_{\Sigma} &= g_{\alpha\beta}\, dx^{\alpha} dx^{\beta} \\ &= g_{\alpha\beta} \left(\frac{\partial x^{\alpha}}{\partial y^i} dy^i \right) \left(\frac{\partial x^{\beta}}{\partial y^j} dy^j \right) \\ &= h_{ij}\, dy^i dy^j \end{aligned}$$

Inverse:

$$h^{ij} e_i^{\alpha} e_j^{\beta} = h^{\alpha\beta} \qquad \text{(ex.)}$$

The <u>extrinsic curvature</u> is:

$$K_{ij} \equiv e_i^{\alpha} e_j^{\beta} \nabla_{\alpha} n_{\beta}$$

extended arbitrarily off Σ

or sometimes

$$K_{\alpha\beta} = \frac{1}{2} \mathcal{L}_n h_{\alpha\beta} = \frac{1}{2} h_{\alpha}{}^{\mu} h_{\beta}{}^{\nu} \mathcal{L}_n g_{\mu\nu}$$

Lie Derivative

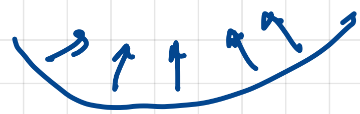
$$\left(\begin{aligned} &\text{for } n \text{ geodesic,} \\ &= \frac{1}{2} \mathcal{L}_n g_{\alpha\beta} \\ &= \nabla_{(\alpha} n_{\beta)} \end{aligned} \right)$$

trace:

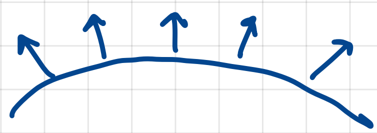
$$K = \underset{h^{ij}K_{ij}}{K^i{}_i} = \underset{g^{\alpha\beta}K_{\alpha\beta}}{K^\alpha{}_\alpha} = \nabla_\alpha n^\alpha$$

Interpretation:

$K < 0$

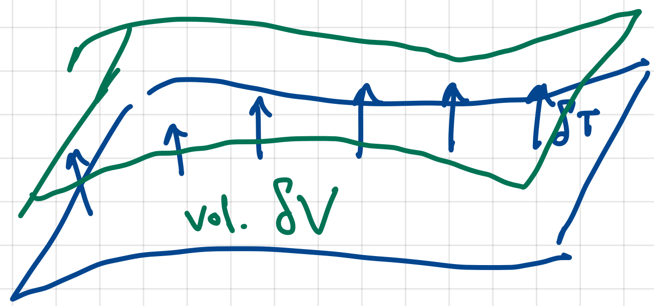


$K > 0$



Usually in GR we discuss intrinsic curvature. $K_{\alpha\beta}$ by contrast is a property of the embedding $\Sigma \to M$, not of the manifold Σ itself.

Claim:



$$K = \frac{1}{\delta V}\frac{d}{d\tau}\delta V$$

$$= \frac{1}{\sqrt{h}}\frac{d}{d\tau}\sqrt{h}$$

(Sort of obvious from the picture!)

Proof: Choose coordinates locally

$$ds^2 = -d\tau^2 + h_{ij}(\tau)\, dy^i dy^j$$

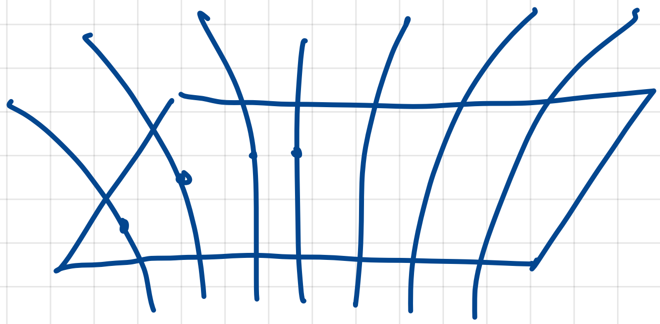
$$n = \partial_\tau$$

$$\frac{1}{\sqrt{h}}\frac{d}{d\tau}\sqrt{h} = \frac{1}{2} h^{ij}\,\partial_\tau h_{ij}$$

$$= \frac{1}{2} h^{ij}\,\mathcal{L}_n h_{ij}$$

$$= K \quad \checkmark$$

Extend n^α off Σ along geodesics



This is a special example of a "geodesic congruence."

$\underbrace{\text{special}}_{}$ ↳ hypersurface orthogonal

(This is a special kind of congruence because it was defined to be hypersurface-orthogonal. This sets twist $\omega_{\alpha\beta} = 0$ in Wald.)

"Expansion"

$$\Theta \equiv \nabla_\alpha n^\alpha = K$$

$$= \frac{1}{\sqrt{h}} \frac{d}{d\tau} \sqrt{h}$$

Raychaudhuri Equation

geodesic deviation $\rightarrow$

$$\frac{d\Theta}{d\tau} = -\frac{1}{D-1}\Theta^2 - \underbrace{\sigma^{\alpha\beta}\sigma_{\alpha\beta}}_{\text{"shear"}} - R_{\alpha\beta}n^\alpha n^\beta$$

$$\sigma_{\alpha\beta} = \nabla_{(\alpha}n_{\beta)} - \frac{1}{D-1}h_{\alpha\beta}$$

(for a hypersurface-orthogonal timelike geodesic congruence)

Focusing theorem

Assume "strong energy condition":

$$\left(T_{\mu\nu} - \frac{1}{2}T g_{\mu\nu}\right) n^\mu n^\nu \geq 0 \quad \text{for timelike } n^\mu$$

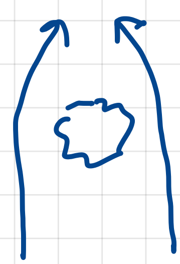
EE $\Rightarrow$

$$R_{\mu\nu}n^\mu n^\nu \geq 0$$

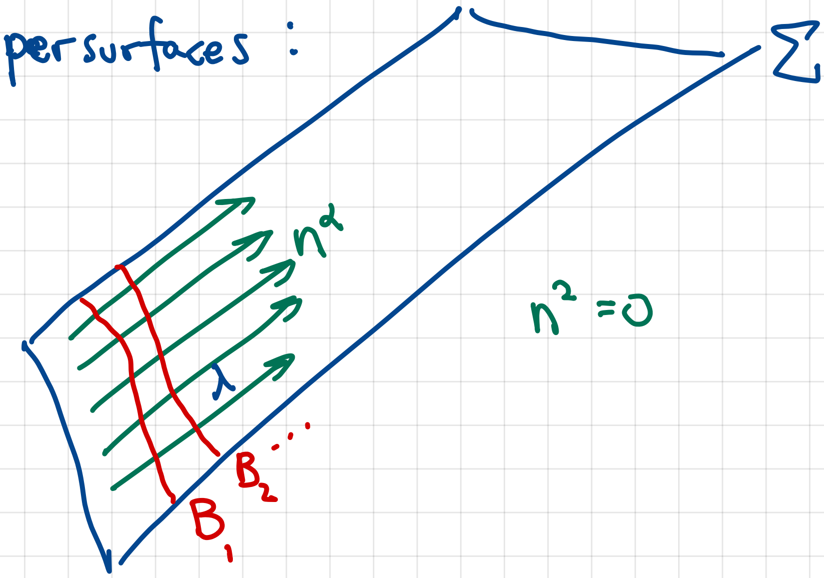
Then

$$\boxed{\frac{d\Theta}{d\tau} \leq 0}$$

"attraction"



Null hypersurfaces:



$$\Theta = \nabla_\alpha n^\alpha$$

$$= \frac{1}{\delta A} \frac{d}{d\lambda} \delta A$$

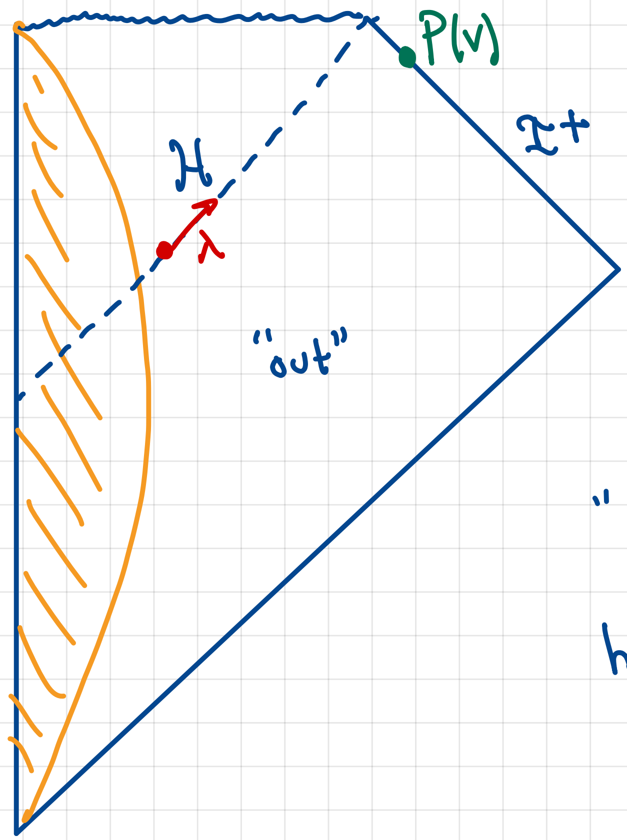
Assuming the "null energy condition",

$$T_{\mu\nu} n^\mu n^\nu \geq 0 \quad \text{for forward-null } n^\mu$$

we get null focusing:

$$\frac{d\Theta}{d\lambda} \leq 0$$

Event Horizon

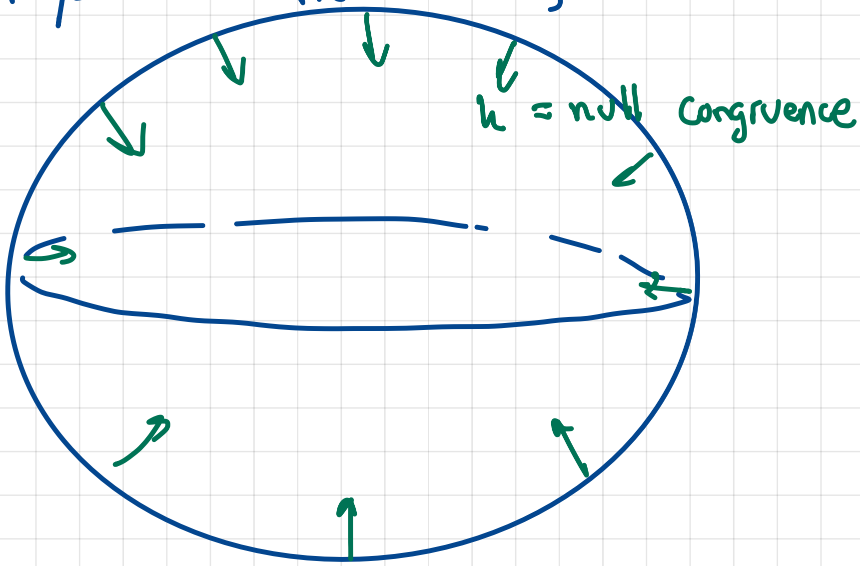


"out" $\equiv$ causal past of $\mathcal{I}^+$

horizon $\mathcal{H} \equiv \partial(\text{"out"})$

Area Theorem

Shoot null rays back from $P(v)$, take $v \to \infty$



These generate the horizon.

$$\Theta = \nabla_\alpha k^\alpha \to 0 \quad \text{as} \quad \lambda \to \infty$$

$$\frac{d\theta}{d\lambda} \leq 0$$

$\Rightarrow \Theta$ is monotonically decreasing to 0

$\Rightarrow \Theta \geq 0$

Along the horizon,

$$\frac{d}{d\lambda} \text{Area} = \frac{d}{d\lambda} \int \sqrt{h}$$

$$= \int \sqrt{h}\, \Theta$$

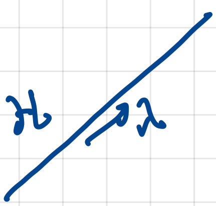
$$\geq 0$$

* The area theorem was stated by Penrose and proved by Hawking.

* It assumed cosmic censorship. Otherwise, "piece of $\mathcal{H}$" could termiate on singularity and never hit $\mathscr{I}^+$.

<u>Evaporation</u>: $\frac{d}{d\lambda}$ Area < 0

$\Rightarrow$ Hawking radiation violates NEC



$T_{\lambda\lambda} \sim -(\text{evap. rate})$